The Shape of Helically Creased Cylinders

K. A. Seffen[1]
e-mail: kas14@cam.ac.uk

N. Borner

Department of Engineering,
University of Cambridge,
Trumpington Street,
Cambridge CB2 1PZ, UK

Creasing in thin shells admits large deformation by concentrating curvatures while relieving stretching strains over the bulk of the shell: after unloading, the creases remain as narrow ridges and the rest of the shell is flat or simply curved. We present a helically creased unloaded shell that is doubly curved everywhere, which is formed by cylindrically wrapping a flat sheet with embedded fold-lines not axially aligned. The finished shell is in a state of uniform self-stress and this is responsible for maintaining the Gaussian curvature outside of the creases in a controllable and persistent manner. We describe the overall shape of the shell using the familiar geometrical concept of a Mohr's circle applied to each of its constituent features—the creases, the regions between the creases, and the overall cylindrical form. These Mohr's circles can be combined in view of geometrical compatibility, which enables the observed shape to be accurately and completely described in terms of the helical pitch angle alone. [DOI: 10.1115/1.4023624]

Keywords: crease, shell, doubly curved, Mohr's circle

1 Introduction

Very thin plates or shells can become creased during excessive out-of-plane deformation. Crumpling a sheet of paper by hand into a ball is a powerful yet simple demonstration: a few creases initially form along lines between sharp points underneath the fingertips so that overall sheet-folding can proceed; this, in turn, allows more of the paper to be manually compressed, thereby forming more, often intersecting, creased lines. Kinematically, creases admit large deformations for a relatively small energy penalty; in particular, they focus, or concentrate, the stretching energy component, which is disproportionately higher than for bending in very thin shells. Correspondingly, the rest of the sheet deforms in bending only so that the creases divide regions that are developable—either singly curved or flat facets [1]. The original complexity of the overall shape can now be rendered through simpler constituent features and these are evinced more clearly when the loading and boundary conditions are simple and carefully controlled. For example, when a sheet is placed on a circular rim and loaded centrally by a downwards point-force, it forms a unique doubly-conical shape everywhere—a "d-cone"—while a single crease grows underneath the loading point and between each conical region [2].

Understanding the characteristics of creased shells over a range of geometrical scales is an active research field in several disciplines. In the physical world, it informs upon many natural processes, including the confined deformation of thin material layers such as films on compliant substrates [3], the draping of thin sheets [4], and the folding/deployment of leaves, insect wings, etc. [5]. Packaging engineers have been folding and shaping flat cardboard sheets into container boxes and sheet metal workers are beginning to pay attention because of the potential reduction in the manufacturing "carbon footprint", compared to traditional methods of rolling and pressing metallic panels.

In this study, we are concerned with the shape of a closed shell that has been "precreased" by folding a flat sheet along straight lines so that it can be wrapped into a cylindrical form. The fold-lines are kinematically identical to creases in that they concentrate the curvature across them; however, because of the way in which the opposing edges of an initially flat sheet are connected, the "interlineal" regions between creases are not developable, as might be expected, but instead are doubly curved. Such folding and wrapping produces a distinctive surface texture and the resulting cylinder is stiffer, both axially and radially, compared to a smooth cylinder under loads applied simply by hand: these stiffnesses are not formally quantified here but we surmise they are similar to the structural benefits eschewed in an earlier study about forming a cube by folding up a flat sheet [6] and may provide another way of improving the performance of practical shells without adding material volume. Instead, we focus on describing the evolution of the final shape of the shell in terms of the interaction between the overall cylindrical shape, the local curvature afforded by the creases, and the influence of the helical pitch; in particular, we assert and confirm that the doubly-curved shape of the interlineal regions, which resemble "strips," is found by subtracting the surface effect of creases from a smooth cylinder; this latter point is a novel aim of study. First, we describe their manufacture from everyday card and how to simplify the assessment of all elements of shape, which then feeds into Sec. 3, where a kinematical analysis is carried out using the familiar concept of a Mohr's circle of curvatures. Section 4 concludes with a final discussion.

2 Manufacture and Measurements

A set of parallel hinge-lines is formed by gently scoring heavy A4 paper-card with a pointed steel scribe so that all lines are inclined at an angle α to either short edge of the sheet. The distance between the lines is fixed at 15 mm but it can be varied around this value by a few millimeters, provided that the spacing is not too wide so that local buckling occurs, leading to nonuniform distortions, or not too narrow so that making accurate measurements becomes difficult. Bending the sheet about the axis of each line requires no effort as the hinge-lines are virtually frictionless, and this forms a freely-wrapped helical shape of interconnected flat facets; see Fig. 1(*a*). The short edges are then connected together as if forming a right-circular cylinder from a smooth sheet, but the sheet has to be gently coerced against the free helical form in a shearing fashion so that the top and bottom edges now lie in a plane and any overlapping hinge-lines are made to coincide for circumferential uniformity. Because of the forcing, the hinge-lines are no longer straight or parallel, but follow identical helical paths now inclined at an angle of pitch α to the new axis of the cylinder. As shown in Fig. 1(*b*), the helices are left-handed when α rotates in the anticlockwise direction and *vice versa*. The surface of the cylinder is smooth between the hinge-lines however, a sharp rotation across them remains, which produces a creased cylinder overall: henceforth, we designate the hinge-lines in this form as creases. The intrinsic coordinates along and normal to each crease are x and y, respectively, and a second angle, β in Fig. 1(*b*), is defined momentarily. A typical constructed cylinder is shown in Fig. 1(*c*).

The detailed shape of the strips between the creases is captured in Fig. 2 in three close-up views. First, Fig. 2(*a*) shows that the strips are not curved widthwise and lines gently drawn onto the surface in this direction evidently remain straight. There is a second direction within each strip along which surface lines are also straight and this is located by carefully rotating a straight edge, such as a ruler, placed normally to the surface until contact is made everywhere while taking care not to indent the strip, and

[1]Corresponding author.

Manuscript received March 6, 2012; final manuscript received January 31, 2013; accepted manuscript posted February 11, 2013; published online July 12, 2013. Assoc. Editor: George Kardomateas.

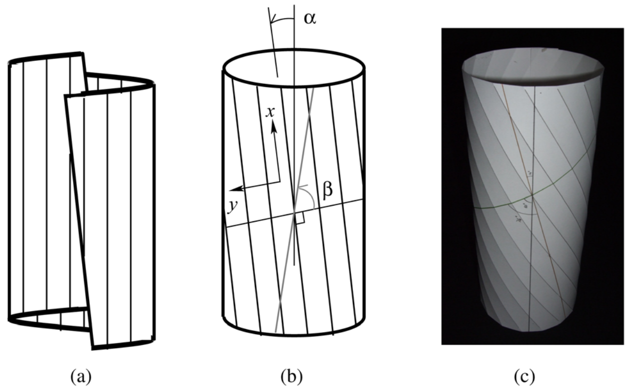

Fig. 1 Manufacture of a creased cylinder. (*a*) Parallel fold-lines are introduced first into a flat rectangular sheet at an angle, α to one of the edges. The sheet then wraps into a helicoidal form when simply bent about each fold-line. (*b*) The opposite short edges of the sheet are connected to form a right-circular cylinder overall with the top and bottom edges each being planar. The angle of pitch of the original fold-lines to the axis of cylinder is α and the coordinates *x* and *y* denote the directions, respectively along and across the interlineal region of strip between the fold-lines, which are now designated as creases. The angle β defines the direction of a line relative to the strip width direction whose properties are discussed in Fig. 2(*c*). A practical cylinder made of A4 paper-card with fold-lines 15 mm apart and inclined at $\alpha = 30°$. A description of the drawn lines is also given in Fig. 2.

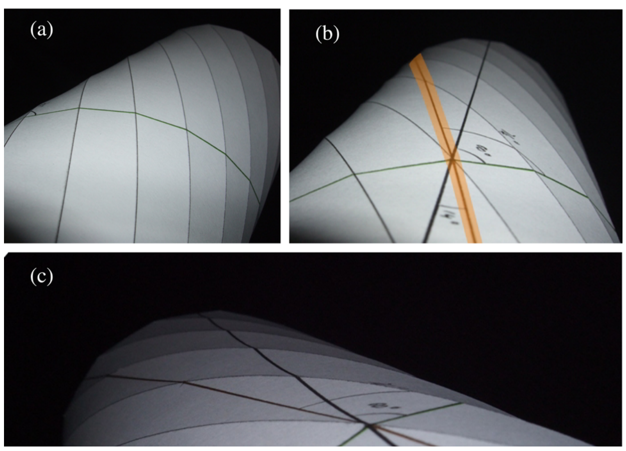

Fig. 2 Views relating to certain properties of the shape, which are amplified by the drawn lines. (*a*) In a direction widthwise and normal to the creased lines, the strip has no surface curvature as given by the straight pencil line: this is repeated over several strips. (*b*) A second direction of zero surface curvature, defined by β in Fig. 1, is found by rotating a flat narrow edge on the surface of each strip region until the edge makes contact everywhere on the strip. This is repeated over all strips in sequence and highlighted to show the contiguous line. (*c*) An axial line on the surface shows some gentle undulation and, hence, some (negative) Gaussian curvature. The rotation across each crease is clearly visible.

then drawing a line along this edge. One particular line is contiguously created in Fig. 2(*b*) by drawing along the straight edge for each strip and using the same end-points for adjacent strips: if the cylinder were laid open into the original flat sheet, this line makes an angle β to the y-axis, which is taken to be positive in the same sense of α. Finally, when a surface line is drawn parallel to the cylinder axis, it is gently curved in the radial direction; see Fig. 2(*c*). In this view, the center of curvature lies outside (and above) the cylinder, in opposition to the hoopwise curvature, and this clearly shows that the strips are doubly curved of negative Gaussian curvature.

These properties are uniform in each strip, i.e., they do not depend on the location within the strip and they are repeated in cylinders with different pitch-angles, where the value of β is found by the same method previously described and recorded against α for several strips before being reported later (in Fig. 5). Uniformity also ensures that the detailed shape can be compactly described; for example, using a Mohr's circle of twisting curvature (or "twist") versus curvature [7] because it expresses directional properties—where the two directions of zero surface curvature can be usefully employed. To complement our understanding of the effect of creasing upon a smooth cylinder, we may think of each crease locally absorbing surface curvature with the overall cylindrical compatibility being upheld; in other words, the original distributed curvature becomes concentrated in each crease, where the closed surface is now discretely shaped. As a corollary, we can find the detailed shape of strips by subtracting the creasing effect, expressed as a surface property, from the originally smooth cylinder and this may be addressed by ascribing separate Mohr's circles for each feature. This approach is novel since it confers surface properties upon the crease even though it is treated as having zero width locally, but also because combining constituent Mohr's circles is, in itself, generally not reported. The latter turns out to be a trivial exercise, provided that the properties concerned have the same local direction, and we demonstrate this first for the more familiar case of an element in plane stress, where adding Mohr's circles is tantamount to adding forces: combining geometrical features in creased cylinders is analogous if, conceptually, less intuitive, and this is performed afterwards.

3 Analysis

Figure 3 revises the procedure for adding together two general states of stress at an elemental point in a structure using Mohr's circles. A reference direction is first declared, which can be the coordinate system for either state. When both states are expressed in the same coordinate system, the corresponding stresses and the shear stress on identical elemental faces are added together; the figure also tracks the addition of vectors for each state, starting at the axes origin and finishing at a pair of stress coordinates, in order to confirm the construction of the third and final Mohr's circle.

Distinguishing each of the surface features of the creased cylinders begins with a Mohr's circle of twisting curvature $\bar{c}$ versus curvature c in Fig. 4(*a*) for a smooth cylinder, which we declare to be the "global" case since it encompasses strips and creases as distinct elements. The principal values of the curvature are zero and κ, respectively, along and circumferentially around the cylinder, whose wrapped radius is $1/\kappa$. Recall that the local direction along the strips x is inclined at α to the axis, thus the principal diameter is rotated by 2α in the same sense for curvature properties of this orientation. The plotting convention is similar to the case for plane stress, in which the end-points are taken to be the coordinates $(\kappa^{G}_{xx}, -\kappa^{G}_{xy})$ and $(\kappa^{G}_{yy}, \kappa^{G}_{xy})$, where the superscript "G," denotes a global property. From simple trigonometry, this leads to the following expressions

$$\kappa^{G}_{xx} = \kappa \sin^2 \alpha, \quad \kappa^{G}_{yy} = \kappa \cos^2 \alpha, \quad \kappa^{G}_{xy} = \kappa \sin\alpha \cos\alpha \qquad (1)$$

The strips are locally flat across their widths and, denoting their properties by a superscript "S," the surface curvature in that direction κ^{S}_{yy} is zero even though, globally, the cylindrical curvature is generally not zero. In view of the original flat sheet, this absence, or "relief," of curving is compatible with the global cylindrical shape only because the creases afford rotation across them; they do not otherwise affect the global properties because they can conform to the global shape by being helically wrapped. Thus, the other curvatures of the strip are equal to the global cylindrical

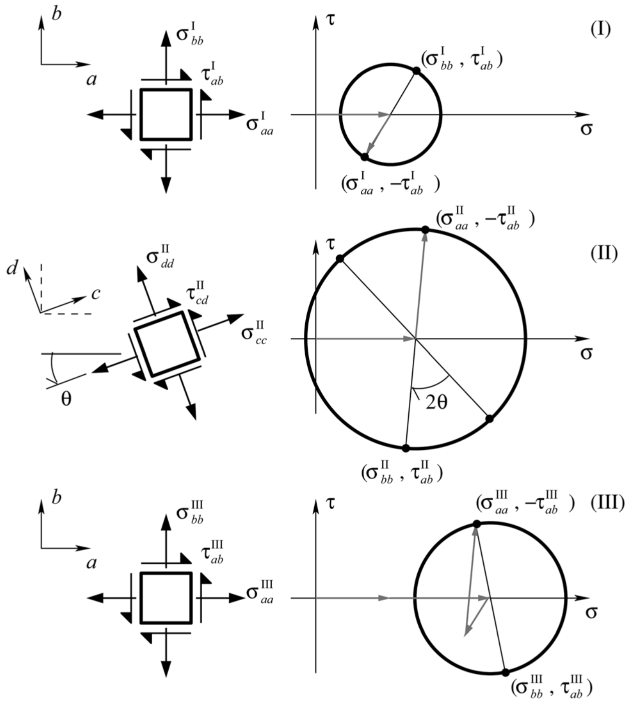

Fig. 3 Addition of Mohr's circles for an element in plane-stress equilibrium. Direct stresses are denoted by σ and shear stresses by τ. Top (I): a general state of stress associated with a coordinate system defined by the $(\boldsymbol{a},\boldsymbol{b})$ axes. The Mohr's circle, which is plotted in (τ,σ) space, shows the diameter formed by the indicated end point coordinates, which follows the usual plotting convention for Mohr's circles familiar to many undergraduates. Also indicated are two vectors to reach the lower end point from the origin by way of the center of the circle. Middle (II): a second state of stress associated with a new set of axes $(\boldsymbol{c},\boldsymbol{d})$, which is rotated from $(\boldsymbol{a},\boldsymbol{b})$ by an arbitrary angle θ. The corresponding diameter (not labeled) is then rotated by 2θ in the opposite direction, to yield the new, and extra, stresses associated with the original direction. Bottom (III): The stress-states from (I) and (II) are combined into a third Mohr's circle by adding the values of stress associated with the same direction: $\sigma_{aa}^{III} = \sigma_{aa}^{I} + \sigma_{aa}^{II}$, $\sigma_{bb}^{III} = \sigma_{bb}^{I} + \sigma_{bb}^{II}$, and $\tau_{ab}^{III} = \tau_{ab}^{I} + \tau_{ab}^{II}$. The pair of vectors from (I) and (II) also combine faithfully to yield the designated end point.

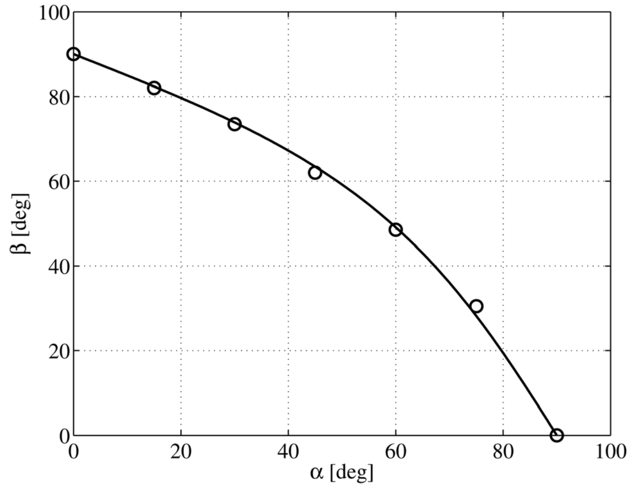

Fig. 5 Predictions (circles) from Eq. (2) compared to measurements for a range of pitch angles of crease. Angle β refers to the direction along which there is zero surface curvature in each of the interlineal strips. Each cylinder is constructed and folded from A4 paper-card, as described in Fig. 1.

properties, namely, the curvature along the strips κ_{xx}^{S} and the twisting curvature κ_{xy}^{S} must be the same as κ_{xx}^{G} and κ_{xy}^{G} in Eq. (1). Consequently, the second direction of zero surface curvature in each strip can be straightforwardly quantified by drawing a Mohr's circle for the strips alone from a diameter with end-points $(\kappa_{xx}^{S}, -\kappa_{xy}^{S}) = (\kappa\sin^2\alpha, -\kappa\sin\alpha\cos\alpha)$ and $(\kappa_{yy}^{S}, \kappa_{xy}^{S}) = (0, \kappa\sin\alpha\cos\alpha)$; see Fig. 4(*b*). Recall from Fig. 2 that there is a second measurable direction defined by angle β in which the strip is flat other than across it and this is predicted by Fig. 4(*b*) by rotating the diameter by 2β in the same sense of α so that the end point of the original κ_{yy}^{S} curvature becomes zero again. Using simple geometry, Fig. 4(*b*) shows that

$$\tan\beta = \frac{\kappa_{xy}^{S}}{\kappa_{xx}^{S}/2} = \frac{2}{\tan\alpha} \tag{2}$$

which is independent of κ. Figure 5 compares this prediction with measurements, and the correlation is very close with a maximum

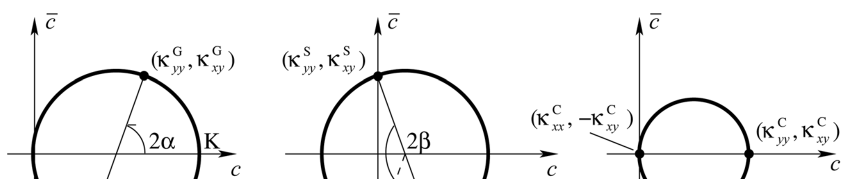

Fig. 4 Description of the surface shape of a creased cylinder using Mohr's circles of twisting curvature (or "twist") $\bar{c}$ versus the curvature c. (*a*) Performance of a smooth right-circular cylinder of radius $1/\kappa$. The principal diameter with end-points (0,0) and $(\kappa, 0)$ is rotated by 2α in the same sense as the fold-lines in Fig. 1, to reveal the local curvatures κ_{xx}^{G} and κ_{yy}^{G}, respectively, along and normal to the creased lines and the associated twist κ_{xy}^{G}. Simple geometry gives: $\kappa_{xx}^{G} = \kappa\sin^2\alpha$, $\kappa_{yy}^{G} = \kappa\sin^2\alpha$, and $\kappa_{xy}^{G} = \kappa\sin^{\alpha}\cos\alpha$. (*b*) Mohr's circle for the strip ("S") regions outside of the creases. The curvature along κ_{xx}^{S} and the twist κ_{xy}^{S} are the same as in (*a*) but the curvature across κ_{yy}^{S} is zero because the strips are flat in this direction. After rotating the diameter by 2β, a second direction of zero surface curvature is found, which corresponds to the measurements highlighted in Fig. 2(*c*). Finally, the equivalent Mohr's circle for a crease is obtained by subtracting (*b*) from (*a*) under the rules applied in Fig. 3. The resulting end-points always lie at the end of a principal diameter: there is no curvature along, so $\kappa_{xx}^{C} = 0$ and there is no twist $\kappa_{xy}^{C} = 0$; the curvature across is $\kappa_{yy}^{C} = \kappa\sin^2\alpha$.

difference of two degrees, which is sufficiently accurate, given that all measurements were carried out by hand-and-eye alone. These results also validate our assumptions about the effect of creasing, which may be identified formally by subtracting the Mohr's circle for the strips from the global case. As in the plane-stress case, each diameter in Figs. 4(*a*) and 4(*b*) conform to the same local orientation, so we may simply subtract the strip curvatures from the global ones for each curvature parameter. The resulting end-points of a diameter for a new Mohr's circle of the crease by itself is reproduced in Fig. 4(*c*) and this describes the equivalent surface properties of the crease in view of how it affects a surface: it does not describe the helical properties associated with the line of crease. As expected, the crease is always rendered as a principal diameter because it cannot confer surface twist or curvature along its length; however, there is an equivalent crosswise curvature κ_{yy}^{C} equal to $\kappa \cos^2 \alpha$, which is the same as that in a smooth cylinder in the same direction. In other words, each crease concentrates the curving required to enable the overall cylindrical form when there is no curving across each strip. Note that we do not need to measure actual curvatures since κ does not feature in Eq. (2) and the behavior of the cylinders here only depends on the crease pitch angle.

4 Discussion

Our creased cylinders are simple to manufacture from card and are doubly curved, unlike the majority of creased structures in the research literature, because of self-stress. Original sets of parallel lines for seeding creases ensure that the final shape is uniform everywhere: the specific properties are fixed only by the orientation within the shell and not by the position. Uniformity allows for a compact description of the shape, which naturally leads to a Mohr's circle approach because it efficiently deals with the directional properties. Other studies have utilized Mohr's circles in the description of folded surfaces but only where the fold-line connects developable regions [8]. Conveniently, we have been able to distinguish the contributions of the creases and the interlineal strips as separate Mohr's circles in themselves and this presents a novel "physical" corollary to how the cylinders were originally made: we start with a smooth cylinder and think of the process of orderly creasing as focusing the developable properties of the cylinder in this direction into hinge-lines, thereby obviating Gaussian curvature everywhere else. The response of other shells is antithetical since creasing relieves the Gaussian curvature ordinarily demanded of the shell during large deformations. Furthermore, we can endow the lineal crease with surface properties, thereby distributing curvature around the cylinder—in the same well-known way that Gaussian curvature is discretely embedded in the vertices of polyhedral surfaces [7].

We also propose that creased shells offer structural benefits in terms of added stiffness and strength and this is due to the surface texture, which creates extra structural depth and to the presence of self-stress in the following way. As noted in Sec. 1, one comparable study involves forming a cube by folding before connecting originally flat faces about straight "ridges"—creases [6]. From the numerical simulations, the authors find that the assembled faces are gently curved and the folding ridges between faces concentrate both the folding and the energy stored in the cube. Indeed, they argue that it is the degree of stored strain energy that underpins the "anomalous strength" of the cube, which emerges from simulated loading. In structures parlance, there is a state of elastic self-stress within the cube before loading and this governs the shape outside of the creased edges: strictly speaking, its variation controls both the overall stiffness and the strength of the cube by offsetting the onset of plasticity and failure. Because there is a higher density of creases in our cylinder compared to the cube, the interlineal regions are relatively narrow and it is possible to confer a higher prestress without inducing local buckling. Consequently, creased cylinders can be as stiff and strong as smooth ones of the same size but with a reduced wall-thickness, which offers potential material savings. For practical materials such as metals, creases from folding will not be frictionless because the material is much stiffer, even for a thin-walled sheet. In addition, the rotations are likely to exceed the elastic limit of the material and residual bending stresses will be focused along the original fold-lines: unless the fold angle before wrapping exactly matches the angle furnished by the crease afterwards (equal to the crease curvature times the strip width), the strip will not be flat across its width, which will affect the other curvature properties. These, and other aspects of study, continue.

Acknowledgment

The insightful comments of two anonymous referees were gratefully received; and to C.F. for good company.

References

[1] Ben Amar, M., and Pomeau, Y., 1997, "Crumpled Paper," Proc. R. Soc. London, Ser. A, **453**, pp. 729–755.
[2] Cerda, E., Chaleb, S., Melo, F., and Mahadevan, L., 1999, "Conical Dislocations in Crumpling," Nature (London), **401**, pp. 46–49.
[3] Cai, S., Breid, D., Crosby, A. J., Suo, Z., and Hutchinson, J. W., 2011, "Periodic Patterns and Energy States of Buckled Films on Compliant Substrates," J. Mech. Phys. Solids, **59**, pp. 1094–1114.
[4] Cerda, E., Mahadevan, L., and Pasini, J. M., 2004, "The Elements of Draping," Proc. Natl. Acad. Sci., **101**, pp. 1806–1810.
[5] Mahadevan, L. and Rica, S., 2005, "Self-Organised Origami," Science, **307**, pp. 1740.
[6] DiDonna, D. A., and Witten, T. A., 2001, "Anomalous Strength of Membranes With Elastic Ridges," Phys. Rev. Lett., **87**, pp. 206105.
[7] Calladine, C. R., 1983, *Theory of Shell Structures*, Cambridge University Press, Cambridge, England.
[8] Duncan, J. P., and Duncan, J. L., 1982, "Folded Developables," Proc. R. Soc. London, Ser. A, **383**, pp. 191–205.